THE BERTINI TRANSFORMATION IN SPACE*

BY

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1. **Introduction.** Examples are known of involutorial transformations I having an invariant pencil of planes through a line l, such that in each plane of the pencil there is a transformation of the Geiser or the de Jonquières type. We have shown in this paper that involutorial transformations exist which have in each plane through l a Bertini transformation with 6 of the fundamental points lying on a C_6, $p=3$, the other two being on l, and either fixed or variable. There is another type in which 6 of the 8 fundamental points lie on a C_6, $p=4$, and in every plane through l there is a degenerate Bertini transformation. A third type is discussed in which there is a net of invariant quartic surfaces through a C_{11}, $p=14$. The method of obtaining this last transformation leads also to an involutorial transformation with a net of invariant surfaces of order $n+1$ through a C_{5n-3} of genus $12n-19$. This type has on each plane through l a Geiser transformation having the 7 fundamental points on C_{5n-3}.

2. **The involutorial Bertini transformation I_B on a cubic surface F_3.** The conics tangent to a cubic surface F_3 at two fixed points O_1, O_2 meet F_3 in two residual points P, P' which are conjugate points of an involutorial Bertini transformation I_B on F_3. The web of quadrics tangent to F_3 at O_1, O_2 meet F_3 in a web of sextic curves of genus 2 which is invariant under I_B as is also the pencil of plane sections through the line $l:O_1+O_2$. If the space (y) of F_3 is transformed into a space (z) by means of the web of cubic surfaces through a fixed C_6, $p=3$, on F_3, then F_3 is transformed into a plane meeting the fundamental sextic of the transformation in 6 points $Q_3, \cdots, Q_8$. If Q_1, Q_2 are the transforms of O_1, O_2, then I_B becomes a plane transformation of order 17, a line going into a $C_{17}:8Q^6$. The image of each six-fold point Q_i is a $C_6:Q_i^3+7Q_j^2$ $(j\neq i)$. The line Q_1Q_2 is the transform of a cubic curve on F_3 through O_1, O_2.

Analytically, if $y_2=0$, $y_1=0$ are the planes tangent to F_3 at $O_1\equiv(1,0,0,0)$, $O_2\equiv(0,1,0,0)$, the equation of F_3 may be written

$$(1)\qquad Ay_1+By_2+C\equiv y_1(a^2y_1y_2+\alpha)+y_2(b^2y_1y_2+\beta)+\gamma y_1y_2+\delta=0,$$

where $\alpha, \beta, \gamma, \delta$ are binary forms in y_3, y_4. The transformation I_B is defined by

$$(2)\qquad Ay_1'=By_2,\ By_2'=Ay_1,\ y_3'=y_3,\ y_4'=y_4.$$

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The images of O_1, O_2 are the sextics in which the quadrics $A=0$, $B=0$ meet F_3. Since the second polars of O_1, O_2 differ from A, B by terms containing y_2, y_1 respectively, the quadrics have second-order contact with F_3 at O_1, O_2 respectively and meet F_3 in sextics having triple points at O_1, O_2 respectively.

3. **The transformation I_B for a pencil of cubic surfaces.** Since a point P determines an F_3 of the pencil (parameter λ) on which P' can be found by the method of §2, we can define involutorial space transformations by either taking O_1, O_2 as fixed points lying on each F_3 of the pencil, or by taking one or both of them variable (the coordinates being functions of λ), on a rational curve lying on each F_3.

4. **Case I, O_1, O_2 are fixed.** For this case we take a pencil of cubic surfaces F_3 having in common a C_9, $p=10$, through $O_1\equiv(1, 0, 0, 0)$, $O_2\equiv(0, 1, 0, 0)$, and write the equation of the F_3 in the form

$$
\begin{aligned}
F_3 &\equiv F_3' - \lambda F_3'' \\
&\equiv ax_1^2x_2 + bx_1x_2^2 + cx_1^2 + dx_1x_2 + ex_2^2 + fx_1 + gx_2 + h = 0,
\end{aligned}
\tag{3}
$$

where $a\equiv a'-\lambda a''$, etc., and c', c'', etc., are binary forms in x_3, x_4. A change of coordinate system given by

$$
y_1 = bx_1 + e, \quad y_2 = ax_2 + c, \quad y_3 = x_3, \quad y_4 = x_4 \tag{4}
$$

will express F_3 in the form (1). The transformation (2) in terms of x_i is

$$
\begin{aligned}
x_1' &= (ax_2B + cB - eA)Ba, \\
x_2' &= (bx_1A - cB + eA)Ab, \\
x_3' &= x_3ABab, \; x_4' = x_4ABab.
\end{aligned}
\tag{5}
$$

The surface of invariant points $K\equiv y_1A-y_2B=0$ contains ab as a factor as does the transformation (5). The forms A, B are of degree 4 in λ and of degree 2 in x_i, so that the x_i' are of degree 5 in x_i and of degree 8 in λ. If λ is replaced by F_3'/F_3'' we have an I_{29} in which the image of O_1 is $A=0$, the image of O_2 is $B=0$, and the image of C_9 can be obtained by applying the transformation to an S_{29}. The table of characteristics of the I_{29} is

$$
\begin{aligned}
O_1 \sim F_{14}&: \overset{7}{O_1} + \overset{6}{O_2} + \overset{4}{C_9}, \\
O_2 \sim F_{14}&: \overset{6}{O_1} + \overset{7}{O_2} + \overset{4}{C_9}, \\
C_9 \sim F_{56}&: \overset{28}{O_1} + \overset{28}{O_2} + \overset{15}{C_9}, \\
S_1 \sim S_{29}&: \overset{14}{O_1} + \overset{14}{O_2} + \overset{8}{C_9}, \\
K_{12}&: \overset{6}{O_1} + \overset{6}{O_2} + \overset{3}{C_9}.
\end{aligned}
$$

Every plane through the line $l:O_1+O_2$ cuts from the F_{56} a composite curve of order 56, the 7 components of which are the images of the 7 residual intersections of C_9 with the plane. If O_i $(i=3, \cdots, 9)$ is any one of these 7 points and $6O_j$ $(j=3, \cdots, 9)$ are the others, then

$$O_i \sim C_8: O_1^4 + O_2^4 + O_i^3 + 6O_j^2 \qquad (i, j = 3, \cdots, 9; i \neq j).$$

In each of these planes there is a transformation of the Bertini type of order 29. If it is transformed by a quadratic transformation having O_1, O_2, O_i for fundamental points it becomes the usual Bertini transformation of order 17 with 8 six-fold points at O_1, O_2, $6O_j$.

Since C_9 is of genus 10 there are 11 trisecants of the C_9 which pass through O_1 or O_2. Any one of these 22 lines meets an S_{29} in $14+2\cdot 8=30$ points and therefore lies on the S_{29}. These lines are the fundamental lines of the second species in the I_{29}. The surface R_{42} of trisecants of C_9 contains C_9 as an 11-fold curve. The line l meets R_{42} in 20 points not on C_9 from which trisecants of C_9 may be drawn. In any one of the 20 planes determined by one of these trisecants and l, the 6 residual intersections of C_9 lie on a conic. Each of these 20 conics meets an S_{29} in $2\cdot 14+4\cdot 8=60$ points and therefore lies doubly on the S_{29}. They are the fundamental conics of the second species in the I_{29}. The tangent planes to the pencil of cubic surfaces at O_1 form a pencil of planes through the tangent line to C_9 at O_1. The plane of the pencil which passes through O_2 cuts from the corresponding F_3 a cubic curve with a double point at O_1 and through O_2 and the 6 residual intersections of C_9 with the plane. There is another such cubic curve with the roles of O_1, O_2 interchanged. Each of these cubics meets an S_{29} in $28+14+6\cdot 8=90$ points and hence lies triply on the S_{29}. They are the fundamental cubics of the second species in the I_{29}. There exist then 22 lines, 20 conics, and 2 cubics which are parasitic curves in the involutorial transformation I_{29}.

If the C_9 is composed of a space cubic C_3 through O_1, O_2 and a C_6, $p=3$, $[C_3, C_6]=8$, the surface F_{56} breaks up into an $F_8:O_1^4+O_2^4+C_3^3+C_6^2$, the image of C_3, and an $F_{48}:O_1^{24}+O_2^{24}+C_3^{12}+C_6^{13}$, the image of C_6. If we transform the space (x) into a space (z) by means of the cubic transformation $T_{3,3}:C_6$ the pencil of F_3's becomes a pencil of planes through the line l' which is the transform of C_3. In each plane through l' there is a Bertini transformation of order 17. The transform of the surface F_8 of trisecants of C_6 is C_6', and to O_1, O_2 correspond the points Q_1, Q_2. The characteristics of the I_{29} in the (z) space are

$$Q_1 \sim F_{10}: Q_1^7 + Q_2^6 + l'^4 + C_6'^2,$$
$$Q_2 \sim F_{10}: Q_1^6 + Q_2^7 + l'^4 + C_6'^2,$$

$$\begin{aligned} l' &\sim F_8: Q_1^4 + Q_2^4 + l'^3 + C_6'^2, \\ C_6' &\sim F_{64}: Q_1^{40} + Q_2^{40} + l'^{28} + C_6'^{13}, \\ S_1 &\sim S_{29}: Q_1^{18} + Q_2^{18} + l'^{12} + C_6'^{6}, \\ &K_{12}: Q_1^6 + Q_2^6 + l'^3 + C_6'^3. \end{aligned}$$

In the (x) space in place of the surface $R_{42}:C_9^{11}$ of trisecants of C_9 we have the surface $R_8:C_6^3$ of trisecants of C_6, the surface $R_8':C_3^4+C_6$ of bisecants of C_3 which meet C_6, and the surface $R_{26}:C_3^7+C_6^7$ of bisecants of C_6 which meet C_3. The 22 parasitic lines in the (x) space are (a) the 4 bisecants of C_3 through O_1 which meet C_6, (b) the 4 bisecants of C_3 through O_2 which meet C_6, (c) the 7 bisecants of C_6 through O_1, (d) the 7 bisecants of C_6 through O_2. The lines of types (a), (b) correspond to parasitic conics in the (z) space through Q_1 or Q_2 and meeting C_6' in 5 points. The lines of types (c), (d) correspond to parasitic lines which are bisecants of C_6' from Q_1 or Q_2. The 8 trisecants of C_6' meeting l' are parasitic and correspond in the (x) space to the 8 points $[C_3, C_6]$.

The line l meets the surface $R_8:C_6^3$ in 8 points, hence 8 trisecants of C_6 meet l. Each of the planes determined by l and one of these trisecants meets the F_3 containing the trisecant in a residual conic which is parasitic. The 8 conics go into parasitic cubics in the (z) space which have double points on C_6' and pass through Q_1, Q_2, and 5 points on C_6'. The surface $R_{26}:C_3^7+C_6^7$ is met by l in 12 points, hence 12 bisecants of C_6 meet C_3 and l. In each of the 12 planes determined by these lines and l there is a parasitic conic which corresponds to a parasitic conic in the (z) space through Q_1, Q_2, and 4 points of C_6'. The two parasitic cubics with double points at O_1 or O_2 and through O_2 or O_1 and 6 points of C_6 correspond to similar cubics in the (z) space. The I_{29} in the (z) space has 22 lines, 20 conics, and 10 cubics which are fundamental curves of the second species.

5. **Case II, O_1 is variable on a space cubic curve C_3.** We take a pencil of cubic surfaces (parameter λ) through a space cubic curve C_3 containing the points $O_1 \equiv (1, \lambda^3, \lambda^2, \lambda)$, $O_2 \equiv (0, 1, 0, 0)$, and having the equation

$$(6)\quad F_3 \equiv \begin{vmatrix} (px) & x_1 & x_4 \\ (qx) & x_4 & x_3 \\ (rx) & x_3 & x_2 \end{vmatrix} \equiv F_3' - \lambda F_3'' \equiv (px)H_p + (qx)H_q + (rx)H_r = 0,$$

where $(px) = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$, $p_i = p_i' - \lambda p_i''$, etc., and H_p, H_q, H_r are quadrics through C_3. A point $P(x)$ determines an F_3 of the pencil and a definite point O_1 so that by the construction of §2 we can determine a point $P'(x')$. A change of variables is made by

$$y_1 = p_2x_4 - q_2x_1,$$
$$y_2 = p(x_2 - 2\lambda x_3 + \lambda^2 x_4) - q(x_3 - 2\lambda x_4 + \lambda^2 x_1),$$
$$y_3 = x_3 - \lambda x_4,$$
$$y_4 = x_4 - \lambda x_1,$$

where $p \equiv \lambda P - Q$, $q \equiv \lambda Q - R$, $P \equiv p_1 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda$, etc. The planes $y_1 = 0$, $y_2 = 0$ are the tangent planes at O_2, O_1 respectively, and the planes $y_3 = 0$, $y_4 = 0$ are a pair of planes through the line $l:O_1 + O_2$. The pencil of cubic surfaces is now of the form (1) and the transformation (2) gives I_B.

The equations of the surfaces $A = 0$, $B = 0$, $K = 0$ may be written in terms of x_i and y_i as follows:

$$A \equiv mp[p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4)\{\lambda q(px) - M(qx) + p(rx)\}]$$
$$- y_2[m(y_3 - \lambda y_4)(\lambda q p_2 - Mq_2 + pr_2) + py_4(Qp_2 - Pq_2)] = 0,$$
$$B \equiv mp[m(p_2H_p + q_2H_q + r_2H_r) - y_4\{q_2(px) - p_2(qx)\}]$$
$$- y_1[m(y_3 - \lambda y_4)(\lambda q p_2 - Mq_2 + pr_2) + py_4(Qp_2 - Pq_2)] = 0,$$
$$K \equiv y_1[p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4)\{\lambda q(px) - M(qx) + p(rx)\}]$$
$$- y_2[m(p_2H_p + q_2H_q + r_2H_r) - y_4\{q_2(px) - p_2(rx)\}] = 0,$$

where $m \equiv \lambda p_2 - q_2$, $M \equiv \lambda p + q$. The surfaces $A = 0$, $B = 0$ are of the second degree in x_i and of degrees 16, 10 in λ respectively. The surface $K = 0$ is of the third degree in x_i and of degree 9 in λ.

We transform the space (x) into a space (z) as was done in the latter part of §4. A surface of the web in the (x) space goes into a surface of the web in the (z) space such that in any plane through l' there is a Bertini transformation of order 17. By this I_{17} the image of l' is a $C_5:Q_1 + Q_2 + 6Q^2$ which with l' makes up a $C_6:8Q^2$ that is the plane section of a sextic surface having Q_1, Q_2, C_6' as double elements. This sextic surface is the transform of a quadric surface through C_3 and tangent to F_3 at O_1, O_2. This quadric which is the image of C_3 and is of the sixth degree in λ has the equation

$$pp_2H_p + pq_2H_q + (\lambda p q_2 - \lambda q p_2 + qq_2)H_r = 0.$$

Any plane through l' is invariant under I_B, hence a pencil of surfaces of the web in the (z) space is made up of the pencil of planes through l' together with the image surfaces of l', Q_1, Q_2. Since the image of Q_1 by the I_{17} in a plane through l' is a $C_6:Q_1^3 + Q_2^2 + 6Q^2$ and since $A = 0$ is of order 16 in λ, the image of Q_1 by the I_B in the (z) space is a surface of order $6 + 16 = 22$ on which l' is a 16-fold line with 3 sheets of the surface having contact along l'. The point Q_1 is $16 + 3 = 19$-fold; Q_2 is a $16 + 2 = 18$-fold point; C_6' is a double curve. In the same way we obtain the surfaces corresponding to Q_2 and l', and the invariant surface K. The table of characteristics of the I_{51} is

$$\begin{aligned}
l' \sim F_{12} &: l'^{7+t} + C_6'^{2} + Q_1^{8} + Q_2^{8},\\
Q_1 \sim F_{22} &: l'^{16+3t} + C_6'^{2} + Q_1^{19} + Q_2^{18},\\
Q_2 \sim F_{16} &: l'^{10+2t} + C_6'^{2} + Q_1^{12} + Q_2^{13},\\
C_6' \sim F_{112} &: l'^{76+12t} + C_6'^{13} + Q_1^{88} + Q_2^{88},\\
S_1 \sim S_{51} &: l'^{34+6t} + C_6'^{6} + Q_1^{40} + Q_2^{40},\\
K_{18} &: l'^{9+3t} + C_6'^{3} + Q_1^{12} + Q_2^{12},
\end{aligned}$$

where the coefficient of t indicates the number of fixed tangent planes at a point of l'.

In determining the number of parasitic lines, conics, and cubics the methods in the previous section have to be changed when the variable point Q_1 is involved. There are 7 bisecants of C_6' through Q_2 and 8 trisecants of C_6' meeting l' which are parasitic. In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l'. The number of coincidences in the (λ, μ) correspondence is $15+7=22$, and hence in 22 positions of Q_1 a bisecant of C_6' can be drawn from Q_1 in the plane through l' associated with Q_1. These bisecants are parasitic lines.

There are 4 conics through Q_2 and 5 points of C_6' in planes through l' which are parasitic. In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l'. There are $12+4=16$ coincidences in this (λ, μ) correspondence and therefore 16 positions of Q_1 such that Q_1 and 5 points of C_6' lie on a conic in the plane through l' associated with Q_1. In any plane λ through l' the 15 conics through Q_2 and 4 points of C_6' meet l' in 15 points μ; through any point μ on l' the 12 conics through Q_2 and 4 points of C_6' lie in 12 planes λ through l'. The $15+12=27$ coincidences of this (λ, μ) correspondence determine 27 positions of Q_1 such that Q_1, Q_2, and 4 points of C_6' lie on a conic in the plane through l' associated with Q_1. These 47 conics are all parasitic.

There are 2 values of λ given by $m=0$ for which the tangent plane to F_3 at O_2 contains O_1, and there are 5 values of λ given by $p=0$ for which the tangent plane to F_3 at O_1 contains O_2. In each of these 7 planes there is a cubic with a double point at O_2 or O_1 and passing through O_1 or O_2 and 6 points of C_6. These 7 cubics correspond to 7 similar cubics in the (z) space. In any plane λ through l' there are 6 cubics with a double point on C_6' and through Q_2 and the 5 remaining points of C_6'. These 6 cubics meet l' in 12 points μ.

Through any point μ on l' there are 8 cubics with a double point on C_6' and through Q_2 and the other 5 points of C_6'. These cubics lie in 8 planes λ through l' so that there are $12+8=20$ coincidences in the (λ, μ) correspondence and 20 positions of Q_1 such that Q_1, Q_2, and the 6 points of C_6' lie on a cubic with a double point at one of these latter points. These cubics lie in planes through l' associated with the positions of Q_1. There are then 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{51}.

6. **Case III, O_1, O_2 are both variable on a space cubic C_3.** To illustrate the case where the points O_1, O_2 are variable on a rational curve which is part of the basis curve of a pencil of cubic surfaces we again utilize a rational space cubic. Other rational curves might be considered and other arrangements of the points O_1, O_2 might be used, but the transformations obtained resemble the I_{51} in Case II and the I_{51} derived in the following case.

The points $O_1\equiv(1, \mu^3, \mu^2, \mu)$, $O_2\equiv(1, -\mu^3, \mu^2, -\mu)$, where $\lambda=\mu^2$, lie on the C_3 which with C_6 makes up the basis of the pencil of cubic surfaces F_3 given by (6). A change of coordinate system is made by

$$\begin{aligned} y_1 &= \bar{p}(x_2 + 2\mu x_3 + \mu^2 x_4) - \bar{q}(x_3 + 2\mu x_4 + \mu^2 x_1), \\ y_2 &= p(x_2 - 2\mu x_3 + \mu^2 x_4) - q(x_3 - 2\mu x_4 + \mu^2 x_1), \\ y_3 &= x_2 - \mu^2 x_4, \\ y_4 &= x_3 - \mu^2 x_1, \end{aligned}$$

where $p=\mu P-Q$, $q=\mu Q-R$, $P=p_1+p_2\mu^3+p_3\mu^2+p_4\mu$, etc., and the dashed letters indicate a change of sign in μ. The surface F_3 is now in the form (1) and the involutorial transformation (2) is determined. We have the following expressions for A, B, K written in terms of x_i and y_i for the sake of conciseness:

$$\begin{aligned} A\equiv{}&4\mu^2\overline{M}M[M(PH_p+QH_q+RH_r)-(y_3-\mu y_4)\{-\mu q(px)+M(qx)-p(rx)\}] \\ &+y_2[y_3\{\overline{M}(\mu\overline{P}q-\overline{Q}M+\overline{R}p)+M(-\mu P\bar{q}-Q\overline{M}+R\bar{p})\} \\ &+\mu y_4\{-\overline{M}(\mu\overline{P}q-\overline{Q}M+\overline{R}p)+M(-\mu P\bar{q}-Q\overline{M}+R\bar{p})\}], \\ B\equiv{}&4\mu^2M\overline{M}[\overline{M}(\overline{P}H_p+\overline{Q}H_p+\overline{R}H_r)-(y_3+\mu y_4)\{\mu\bar{q}(px)+\overline{M}(qx)-\bar{p}(rx)\}] \\ &+y_1[y_3\{\overline{M}(\mu\overline{P}q-\overline{Q}M+\overline{R}p)+M(-\mu P\bar{q}-Q\overline{M}+R\bar{p})\} \\ &+\mu y_4\{-\overline{M}(\mu\overline{P}q-\overline{Q}M+\overline{R}p)+M(-\mu P\bar{q}-Q\overline{M}+R\bar{p})\}], \\ K\equiv{}&y_1[M(PH_p+QH_q+RH_r)-(y_3-\mu y_4)\{-\mu q(px)+M(qx)-p(rx)\}] \\ &-y_2[\overline{M}(\overline{P}H_p+\overline{Q}H_q+\overline{R}H_r)-(y_3+\mu y_4)\{\mu\bar{q}(px)+\overline{M}(qx)-\bar{p}(rx)\}], \end{aligned}$$

where $M=\mu p+q$.

The surfaces $A=0$, $B=0$ are of order 2 in x_i and of order 13 in μ^2 after the removal of a factor μ^2. The invariant surface $K=0$ is of order 9 in μ^2 and of order 3 in x_i. The image of C_3 which is the quadric which contains C_3 and is tangent to F_3 at O_1, O_2 is of order 6 in μ^2 and has the equation

$$(\bar{p}M - p\overline{M})H_p + \mu(p\bar{q} + q\bar{p})H_q + \mu(\bar{q}M + q\overline{M})H_r = 0.$$

The table of characteristics of the I_B in the (z) space may be obtained as in Case II and with the same results except that the images of the points Q_1, Q_2 combine and the joint image is

$$(Q_1, Q_2) \sim F_{38}\colon l'^{26+5t} + C_6'^4 + (Q_1, Q_2)^{31}.$$

In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l'. In the correspondence (λ, μ) there are $15+7+7=29$ coincidences since $\lambda=\mu^2$, and hence in 29 positions of the pair of points Q_1, Q_2 a bisecant of C_6' can be drawn from one of them in the plane through l' associated with the pair. There are 8 trisecants of C_6' which meet l'. These 37 lines are parasitic in I_{51}.

In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l'. The number of coincidences in the (λ, μ) correspondence is $12+4+4=20$ and hence in 20 positions of the pair of points Q_1, Q_2, one of the pair and 5 points of C_6' lie on a conic in the plane through l' associated with the pair. In any plane λ there is a pencil of conics through each of the 15 sets of 4 of the 6 points of C_6'. Each pencil determines an involution on l' which has one pair in common with the involution of points μ, hence 15 pairs of points μ^2 are determined. Given any pair of points μ^2 on l' there are 12 planes λ through l' in which there are conics through the pair μ^2 and 4 points of C_6'. In the correspondence (λ, μ^2) the $15+12=27$ coincidences fix 27 positions of the pair Q_1, Q_2 such that conics in the associated planes pass through them and 4 of the points of C_6'.

The 7 values of λ given by $M\overline{M}=0$ determine 7 planes tangent to F_3 at O_1 or O_2 which pass through O_2 or O_1. From the associated F_3 each of these planes cuts a cubic with a double point at O_1 or O_2 and passing through O_2 or O_1 and 6 points of C_6. These 7 cubics correspond to similar cubics in the (z) space which are parasitic in the I_{51}. In any plane λ through l' there are 6 pencils of cubics through the 6 points of C_6' and with a double point at one of them. Each pencil determines an involution of the third order on l' which has 2 pairs in common with the involution of points μ, hence to a λ correspond 12 pairs of points μ^2. Given any pair of points μ^2 on l' there are 8 planes λ

through l' in which there are cubics through the pair μ^2 and 6 points of C_6' and which have a double point at one of the points of C_6'. The correspondence (λ, μ^2) has $12+8=20$ coincidences which determine 20 positions of the pair Q_1, Q_2 such that in the associated plane there will be a cubic through Q_1, Q_2 and the 6 points of C_6' and having a double point at one of the points of C_6'. Hence as in Case II we have 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{51}.

7. **A Bertini transformation on a cubic variety in S_4.** In a space of four dimensions we take a cubic variety V_3 with a double point at $O_5 \equiv (0, 0, 0, 0, 1)$ and through the points $O_1 \equiv (1, 0, 0, 0, 0)$, $O_2 \equiv (0, 1, 0, 0, 0)$. The equation of the variety is

$$V_3 \equiv \phi_2 x_5 + \phi_3 = 0,$$

where ϕ_2, ϕ_3 are quaternary forms in x_1, x_2, x_3, x_4 with the x_1^3, x_2^3 terms missing in ϕ_3. The conics tangent to V_3 at the points O_1, O_2 meet V_3 in two residual points P, P' which are conjugate points in a Bertini involution J_B on V_3. This involution can be mapped on the 3-space $x_5=0$, and a Bertini involution I_B in 3-space is thus determined. The hyperplane $x_5=0$ meets V_3 in the cubic surface $\phi_3=0$, and meets the tangent hypercone to V_3 at O_5 in the quadric $\phi_2=0$. The surfaces $\phi_2=0$, $\phi_3=0$ meet in a sextic curve C_6 of genus 4. Any plane π through the line O_1O_2 meets C_6 in 6 points R which lie on a conic. The hyperplanes through O_1, O_2 are invariant under J_B, and the planes π are invariant under I_B. Since the 6 points R lie on a conic in each plane π, the Bertini involution in such a plane is degenerate and of the form $I_{13}: O_1^6+O_2^6+6R^4$, with an invariant curve $k_7: O_1^3+O_2^3+6R^2$. The I_B in the space $x_5=0$ has the characteristics

$$\begin{aligned}
O_1 &\sim F_6 : \overset{3}{O_1} + \overset{2}{O_2} + \overset{2}{C_6},\\
O_2 &\sim F_6 : \overset{2}{O_1} + \overset{3}{O_2} + \overset{2}{C_6},\\
C_6 &\sim F_{24}: \overset{12}{O_1} + \overset{12}{O_2} + \overset{7}{C_6},\\
S_1 &\sim S_{13}: \overset{6}{O_1} + \overset{6}{O_2} + \overset{4}{C_6},\\
& \quad K_7 : \overset{3}{O_1} + \overset{3}{O_2} + \overset{2}{C_6}.
\end{aligned}$$

The 6 bisecants of C_6 from O_1 and the 6 from O_2 are parasitic lines in I_B and correspond to lines on the V_3 through O_1 or O_2. To determine the number of parasitic conics we must find the number of conics which lie on V_3 and pass through O_1 and O_2, since in any such conic the construction used to determine J_B will fail in the sense that to a point on the conic corresponds the whole conic. By a proper choice of coordinate system we can write the equation of any cubic variety in the form

$$x_1^2x_2 + x_1x_2^2 + ax_1 + bx_2 + cx_1x_2 + d = 0 \tag{7}$$

where a, b, c, d are ternary forms in x_3, x_4, x_5. The left hand member of (7) can be factored as follows:

$$(x_1x_2 + b)(x_1 + x_2 + c),$$

if

$$a - b = 0 \text{ and } ac - d = 0. \tag{8}$$

Equations (8) represent two hypercones of the second and third orders respectively whose rulings are planes. The 6 planes common to the two hypercones cut conics from the cubic variety through the points O_1, O_2. Hence there are 6 fundamental conics of the second species in the I_{13} besides the 12 lines of the second species.

8. **A family of space Bertini transformations.** A net of planes $\pi \equiv \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 = 0$ through the point (0, 0, 0, 1) and a net of cubic surfaces

$$F_3 \equiv x_4(ax)^2 + x_1x_2(bx) \equiv \lambda_1F_3' + \lambda_2F_3'' + \lambda_3F_3''' = 0, \tag{9}$$

where $(ax)^2$ and (bx) are quaternary forms in x_1, x_2, x_3, x_4, and

$$a_{ij} \equiv \lambda_1a_{ij}' + \lambda_2a_{ij}'' + \lambda_3a_{ij}''', \text{ and } b_i \equiv \lambda_1b_i' + \lambda_2b_i'' + \lambda_3b_i''',$$

through the lines $l_1 \equiv x_2 = x_4 = 0$ and $l_2 \equiv x_1 = x_4 = 0$ may be used to determine a transformation of the Bertini type. A point $P(x)$ determines a set of λ_i and hence a plane π and a surface F_3. The plane π cuts the lines l_1, l_2 in a pair of points $O_1(\lambda_3, 0, -\lambda_1, 0)$, $O_2(0, \lambda_3, -\lambda_2, 0)$. The conic through $P(x)$ and tangent to F_3 at O_1 and O_2 will meet F_3 in a residual point $P'(x')$ which is the conjugate of $P(x)$ in an involutorial transformation I.

If we make the linear transformation

$$\begin{aligned} y_1 &= \lambda_3B_2x_1 + A_2x_4, \\ y_2 &= \lambda_3B_1x_2 + A_1x_4, \\ y_3 &= \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3, \\ y_4 &= x_4, \end{aligned}$$

where

$$\begin{aligned} B_1 &\equiv b_1\lambda_3 - b_3\lambda_1, \\ B_2 &\equiv b_2\lambda_3 - b_3\lambda_2, \\ A_1 &\equiv a_{11}\lambda_3^2 - 2a_{13}\lambda_1\lambda_3 + a_{33}\lambda_1^2, \\ A_2 &\equiv a_{22}\lambda_3^2 - 2a_{23}\lambda_2\lambda_3 + a_{33}\lambda_2^2, \end{aligned}$$

then equation (9) is in the form of (1), and the transformation (2) may be used to obtain

$$x_1' = B_1B(By_2 - A_2Ay_4),$$
$$x_2' = B_2A(Ay_1 - A_1By_4),$$
$$x_3' = - B_1B(By_2 - A_2Ay_4) - B_2A(Ay_1 - A_1By_4),$$
$$x_4' = \lambda_3B_1B_2ABy_4,$$

where

$$A \equiv B_1^2y_1y_2 + B_2y_4^2[A_1^2B_2 + 2\lambda_3^2B_1^2(a_{14}\lambda_3 - a_{34}\lambda_1) + A_1B_1(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)],$$
$$B \equiv B_2^2y_1y_2 + B_1y_4^2[A_2^2B_1 + 2\lambda_3^2B_2^2(a_{24}\lambda_3 - a_{34}\lambda_2) + A_2B_2(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)].$$

The λ_i are now replaced by

$$\lambda_1 \equiv \phi_1 \equiv x_2F_3''' - x_3F_3'',$$
$$\lambda_2 \equiv \phi_2 \equiv x_3F_3' - x_1F_3''',$$
$$\lambda_3 \equiv \phi_3 \equiv x_1F_3'' - x_2F_3'.$$

The quartic surfaces $\phi_i = 0$ have in common the lines l_1, l_2, and a residual curve C_{11} of order 11 and genus 14. The surfaces $A=0$, $B=0$ which are the images of the lines l_1, l_2 are of order 8 in ϕ_i and 2 in x_i after a factor ϕ_3^2 is removed. The factor $\phi_3^5B_1B_2$ can be removed from the transformation, and the invariant surface $K \equiv y_1A - y_2B = 0$ has the factor $\phi_3^3B_1B_2$. The characteristics of the transformation are

$$l_1 \sim F_{34} : l_1^{11} + l_2^{10} + C_{11}^{8},$$
$$l_2 \sim F_{34} : l_1^{10} + l_2^{11} + C_{11}^{8},$$
$$C_{11} \sim F_{204}: l_1^{66} + l_2^{66} + C_{11}^{47},$$
$$S_1 \sim S_{69} : l_1^{22} + l_2^{22} + C_{11}^{16},$$
$$K_{27} : l_1^{9} + l_2^{9} + C_{11}^{6}.$$

The x parasitic lines of I are trisecants of C_{11} which meet either l_1 or l_2. Since C_{11} meets l_1 in 4 points there are 7 residual intersections R_i in a plane through l_1. In any such plane a line R_iR_j meets l_1 in a point P, and through each of R_i and R_j pass 5 other bisecants of C_{11} meeting l_1 in 10 points Q. If h' is the number of bisecants of C_{11} through any point of l_1, the points P, Q are in $(10h', 10h')$ correspondence. The $20h'$ coincidences are determined by the x trisecants of C_{11} meeting l_1, the r' tangents of C_{11} meeting l_1, and the 4 tangents to C_{11} where it meets l_1. Hence

$$20h' = 6x + 5r' + 30\cdot 4.$$

Since the C_{11} is of class $r=48$ and has $h=31$ apparent double points, then

$h' = h - 4\cdot 3/2 = 25$, and $r' = r - 2\cdot 4 = 40$. These values make $x = 30$, but among the 30 trisecants the line l_2, which is a quadrisecant, is counted 4 times. Hence there are 26 trisecants of C_{11} meeting l_1 and 26 more which meet l_2. These 52 lines are the parasitic lines of the transformation I.

Let y be the number of parasitic conics and z be the number of parasitic cubics of I. The complete intersection of two surfaces of the web of S_{69} is made up of

$$69^2 = 69 + 22^2 + 22^2 + 11\cdot 16^2 + 52 + 8y + 27z,$$

and the complete intersection of an S_{69} and the K_{27} is made up of

$$69\cdot 27 = 27 + 9\cdot 22 + 9\cdot 22 + 6\cdot 16\cdot 11 + 52 + 4y + 9z.$$

The solution of these equations is $y = 45$, $z = 18$, whence we can conclude that the fundamental curves of the second species in I consist of 52 lines, 45 conics, and 18 cubics.

9. **A family of space Geiser transformations.** If $F_n = 0$ is a surface of order n with an $(n-2)$-fold line $l \equiv x_3 = x_4 = 0$, the equations

(10) $\pi \equiv \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$

(11) $F_n \equiv a x_1^2 + b x_2^2 + 2h x_1 x_2 + 2f x_2 + 2g x_1 + c \equiv \lambda_1 F_n' + \lambda_2 F_n'' + \lambda_3 F_n''' = 0,$

where $a \equiv \lambda_1 a' + \lambda_2 a'' + \lambda_3 a'''$, etc., and a', a'', a''', etc., are binary forms in x_3, x_4, define a net of plane curves C_n of order n with an $(n-2)$-fold point $Q \equiv (\lambda_2, -\lambda_1, 0, 0)$. A line through Q and a point $P(x)$ on C_n meets it in a residual point $P'(x')$, thus defining an involutorial transformation I having the invariant net of surfaces

$$k_1\phi_1 + k_2\phi_2 + k_3\phi_3 \equiv k_1(x_2 F_n''' - x_3 F_n'') + k_2(x_3 F_n' - x_1 F_n''') + k_3(x_1 F_n'' - x_2 F_n') = 0.$$

The pencil of planes $p \equiv x_4 - \mu x_3 = 0$ through l are invariant under I and in any such plane F_n takes the form

(12) $$a x_1^2 + b x_2^2 + c x_3^2 + 2h x_1 x_2 + 2f x_2 x_3 + 2g x_1 x_3 = 0,$$

where the coefficients are polynomials in μ. This net of conics enables us to map I on a double space $S(\lambda_1 : \lambda_2 : \lambda_3, \mu)$. A plane

(13) $$m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = 0$$

is mapped on S by eliminating x_i between (10) and (13) and using $x_4 = \mu x_3$. The values of x_i thus obtained are substituted in (12) giving

$$
\begin{aligned}
&a(m_2\lambda_3 - \bar{m}_3\lambda_2)^2 + b(\bar{m}_3\lambda_1 - m_1\lambda_3)^2 + c(m_1\lambda_2 - m_2\lambda_1)^2 \\
&\quad + 2h(m_2\lambda_3 - \bar{m}_3\lambda_2)(\bar{m}_3\lambda_1 - m_1\lambda_3) + 2f(\bar{m}_3\lambda_1 - m_1\lambda_3)(m_1\lambda_2 - m_2\lambda_1) \\
&\quad + 2g(m_2\lambda_3 - \bar{m}_3\lambda_2)(m_1\lambda_2 - m_2\lambda_1) = 0, \text{ where } \bar{m}_3 \equiv m_3 + \mu m_4,
\end{aligned}
$$

which must be identical with

$$(m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4)(m_1x_1' + m_2x_2' + m_3x_3' + m_4x_4') = 0.$$

From this identity we have

$$
\begin{aligned}
x_1x_1' &= b\lambda_3^2 - 2f\lambda_2\lambda_3 + c\lambda_2^2, \\
x_2x_2' &= c\lambda_1^2 - 2g\lambda_1\lambda_3 + a\lambda_3^2, \\
x_3x_3' &= a\lambda_2^2 - 2h\lambda_1\lambda_2 + b\lambda_2^2, \\
x_4' &= \mu x_3'.
\end{aligned}
$$

If we replace μ by x_4/x_3 and λ_i by ϕ_i we have the transformation I in the form

$$
\begin{aligned}
x_1x_1' &= b\phi_3^2 - 2f\phi_2\phi_3 + c\phi_2^2, \\
x_2x_2' &= c\phi_1^2 - 2g\phi_1\phi_3 + a\phi_3^2, \\
x_3' &= x_3(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2), \\
x_4' &= x_4(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2),
\end{aligned}
$$

where x_1, x_2 are factors of the first two equations respectively. The surfaces $\phi_i = 0$ are of order $n+1$ and have l as an $(n-2)$-fold line. The residual basis curve of the net of ϕ_i is a C_{5n-3} of order $5n-3$ and genus $12n-19$ through the point $(0, 0, 0, 1)$. The image of l in I is the surface $L \equiv a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2 = 0$, which is of order 3 in ϕ_i and of order $n-2$ in x_3, x_4. The image in S of the invariant surface K has the equation

$$
\begin{vmatrix}
a & h & g & \lambda_1 \\
h & b & f & \lambda_2 \\
g & f & c & \lambda_3 \\
\lambda_1 & \lambda_2 & \lambda_3 & 0
\end{vmatrix} = 0,
$$

which corresponds to K^2 in the space (x). Hence K is of order 2 in ϕ_i and of order $n-1$ in x_3, x_4. The table of characteristics of I is

$$
\begin{aligned}
l \sim L_{4n+1} &: l^{4n-7} + C_{5n-3}^3, \\
C_{5n-3} \sim F_{12n+3} &: l^{12n-18} + C_{5n-3}^8, \\
S_1 \sim S_{4n+2} &: l^{4n-6} + C_{5n-3}^3, \\
K_{3n+1} &: l^{3n-5} + C_{5n-3}^2.
\end{aligned}
$$

In any plane p through l there is an ordinary Geiser transformation, therefore the C_{5n-3} meets such a plane in the 7 fundamental points R_i of the Geiser transformation and in $5n-10$ points on l. The section of C_{5n-3} by the plane $x_3=0$ is the point (0, 0, 0, 1) and 6 points lying on the conic $x_3=0$, $F_n{}''' =0$. Hence on this plane the Geiser transformation degenerates and the conic is parasitic for I.

The x parasitic lines of I are trisecants of C_{5n-3} meeting l. Since C_{5n-3} meets any plane p in 7 points not on l the method of §8 may be used in determining the number of trisecants of C_{5n-3} which meet l. The number $x=15n-15$ is obtained from the equation

$$20h' = 6x + 5r' + 30(5n - 10),$$

where $r'=24n-26$, and $h'=18n-26$. Therefore the fundamental curves of the second species for I consist of $15n-15$ lines and one conic.

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